The Hardy-Littlewood type theorems in Besov spaces with the Haar basis

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Abstract: In this paper there were established the coefficient necessary and sufficient conditions that the function belongs to the Besov spaces with the Haar basis.

Let $H = \{h_m(t)\}_{m=1}^{+\infty}$, $t \in [0,1]$ be the orthonormal Haar system [?].

We denote by $\|f\|_p$ the norm of function from $L_p[0,1]$, $1 \le p < +\infty$.

Let $f \in L_p[0,1]$, $1 \le p < +\infty$. Let $E_n(f)_{p\theta}$ be the best approximation of function f in the metrics of Lebesgue space $L_p[0,1]$ by the Haar polynomials of order not higher than n.

Definition 0.1. *Let $1 \le p \le +\infty$, $1 \le \theta \le +\infty$, $r > 0$. We say that function $f \in L_p[0,1]$ belongs to the Besov space with the Haar basis, if*

$$|f; B^r_{p\theta}([0,1]; H)| = \|f\|_p + \left\{ \sum_{k=0}^{+\infty} 2^{kr\theta} E^{\theta}_{2^k}(f)_p \right\}^{\frac{1}{\theta}}, \quad 1 \le \theta < +\infty;$$

$$|f; B^r_{p\infty}([0,1]; H)| = \|f\|_p + \sup\left\{ 2^{kr} E_{2^k}(f)_p \mid k \in \mathbb{Z}^+ \right\}.$$

The following two theorems contain GM-monotone condition on [2] the Fourier-Haar coefficients of function $f \in B^r_{p\theta}([0,1], H)$.

Theorem 0.2. *Let $f \in L[0,1]$. Let $f(x) \sim \sum_{k=1}^{+\infty} a_k h_k(x)$ be its the Fourier-Haar series. If the positive sequence of the Fourier-Haar coefficients $\{a_k\} \in GM$, then for $f \in B^r_{s\theta}([0,1]; H)$, where $r > 0$, $1 < s < +\infty$, $1 \le \theta \le +\infty$, it is sufficient that for some p: $1 \le p < s$*

$$(1) \qquad \left\{ \sum_{k=1}^{+\infty} k^{r\theta + \theta\left(\frac{1}{2} + \frac{1}{p} - \frac{1}{s}\right) - 1} a_k^{\theta} \right\}^{\frac{1}{\theta}} < +\infty.$$

Moreover, the following inequality

$$|f; B^r_{s\theta}([0,1]; H)| \le A_{psr\theta} \left\{ \sum_{k=1}^{+\infty} k^{r\theta + \theta\left(\frac{1}{2} + \frac{1}{p} - \frac{1}{s}\right) - 1} a_k^{\theta} \right\}^{\frac{1}{\theta}}.$$

holds. Here $A_{psr\theta} > 0$ depends only on p, s and r.

Theorem 0.3. *Let $r > 0$, $1 < s < \lambda < +\infty$, $1 \le \theta \le +\infty$. Let $f \in B^r_{s\theta}([0,1]; H)$. If its Fourier-Haar coefficients are positive and GM-monotone, then*

$$|f; B^r_{s\theta}([0,1]; H)| \ge D_{\lambda sr\theta} \left\{ \sum_{k=1}^{+\infty} k^{r\theta + \theta\left(\frac{1}{2}+\frac{1}{\lambda}-\frac{1}{s}\right)-1} a_k^\theta \right\}^{\frac{1}{\theta}}.$$

Here $D_{\lambda sr\theta} > 0$ depends only on λ, s, r and θ.

Keywords: Fourier series, Haar system, Fourier-Haar coefficients, GM-monotone sequences, Hardy-Littlewood theorem, Besov space

2010 Mathematics Subject Classification: 42A16, 46E30

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