On relativistic velocity addition

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Abstract: In Einstein's relativity theory, the sum of the velocities $\mathbf{v}_1 = \langle v_1, 0, 0\rangle$ and $\mathbf{v}_2 = \langle v_2, 0, 0\rangle$ equals to $\mathbf{v} = \langle v, 0, 0\rangle$ with

$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}, \tag{1}$$

where c is the speed of light. This is a special case when the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ have the same direction. In more general case, when $\mathbf{v}_1$ and $\mathbf{v}_2$ have different directions, the formula is more complicated and does not have ordinary properties such as linearity, commutativity, associativity, etc. In this work we are aimed to develop algebraic operations over the preceding relativistic addition formula which have all ordinary properties.

The method that we will use is based on non-Newtonian calculi initiated in [**?**]. Previously, non-Newtonian calculus with the exponential reference function, which has the multiplicative and bigeometric modifications, was studied in [**?**,**?**]. In this paper, a suitable non-Newtonian calculus is with the hyperbolic tangent reference function.

We will operate with relative speeds, that is, instead of v we will consider v/c assuming that $|v| < c$. The relative speeds are numbers in $\mathbb{E} = (-1, 1)$. Then for relative speeds v_1 and v_2, Eq. (**??**) produces the relative speed v by

$$v = v_1 \oplus v_2 = \frac{v_1 + v_2}{1 + v_1 v_2}.$$

This is an addition formula in $\mathbb{E}$.

For other algebraic operations in $\mathbb{E}$, consider hyperbolic tangent function and its inverse defined by

$$\alpha(x) = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}},\ x \in \mathbb{R},$$

and

$$\alpha^{-1}(x) = \tanh^{-1} x = \frac{1}{2} \ln \frac{1 + x}{1 - x},\ x \in \mathbb{E}.$$

It is known that (see, for example, [?])

$$\text{(2)} \qquad \text{(i)} \quad \mathrm{a} \oplus \mathrm{b} = \alpha(\alpha^{-1}(\mathrm{a}) + \alpha^{-1}(\mathrm{b})) = \frac{\mathrm{a}+\mathrm{b}}{1+\mathrm{ab}}.$$

Based on this, introduce the other algebraic operations on $\mathbb{E}$ by

$$\text{(ii)} \quad a \ominus b = \alpha(\alpha^{-1}(a) - \alpha^{-1}(b)) = \frac{a-b}{1-ab},$$

$$\text{(iii)} \quad a \otimes b = \alpha(\alpha^{-1}(a) \times \alpha^{-1}(b)) = \frac{\left(\frac{1+a}{1-a}\right)^{\frac{1}{2}\ln\frac{1+b}{1-b}} - 1}{\left(\frac{1+a}{1-a}\right)^{\frac{1}{2}\ln\frac{1+b}{1-b}} + 1},$$

$$\text{(iv)} \quad a \oslash b = \alpha(\alpha^{-1}(a)/\alpha^{-1}(b)) = \frac{\left(\frac{1+a}{1-a}\right)^{\frac{2}{\ln\frac{1+b}{1-b}}} - 1}{\left(\frac{1+a}{1-a}\right)^{\frac{2}{\ln\frac{1+b}{1-b}}} + 1}.$$

The formulae for $a \otimes b$ and $a \oslash b$ are so that it is difficult to predict them by intuition unless they are not transformed in accordance to hyperbolic tangent reference function.

The set $\mathbb{E}$ with the operations (i)-(iv) transforms all properties of $\mathbb{R}$ to $\mathbb{E}$ making it an ordered field. In particular, the distributivity property can be verified as follows:

$$\begin{aligned}
(a \oplus b) \otimes c &= \alpha(\alpha^{-1}(a) + \alpha^{-1}(b)) \otimes c \\
&= \alpha(\alpha^{-1}(\alpha(\alpha^{-1}(a) + \alpha^{-1}(b))) \times \alpha^{-1}(c)) \\
&= \alpha((\alpha^{-1}(a) + \alpha^{-1}(b)) \times \alpha^{-1}(c)) \\
&= \alpha((\alpha^{-1}(a) \times \alpha^{-1}(c) + \alpha^{-1}(b) \times \alpha^{-1}(c)) \\
&= \alpha((\alpha^{-1}(a) \times \alpha^{-1}(c)) \oplus \alpha(\alpha^{-1}(b) \times \alpha^{-1}(c)) \\
&= (a \otimes c) \oplus (b \otimes c).
\end{aligned}$$

Moreover, the vector operations (addition, product by scalar, scalar product, vector product, etc.), defined through operations (i)-(iv) of $\mathbb{E}$ have all properties of the same operations defined over $\mathbb{R}$.

Whole ordinary calculus transforms to a new calculus that we can call tanh-calculus, in which the sense of derivative and integral changes. If $f^\star$ denotes the tanh-derivative of f, then

$$\text{(3)} \qquad f^\star(x) = \alpha\left(\frac{\alpha^{-1}(f(x))'}{\alpha^{-1}(x)'}\right) = \frac{e^{\frac{2f'^{2})}{1-f^2(x)}} - 1}{e^{\frac{2f'^{2})}{1-f^2(x)}} + 1}.$$

This can be Informally derived as follows:

$$\begin{aligned}
((f(y)\ominus f(x))\oslash(y\ominus x)) &= \alpha\left(\frac{\alpha^{-1}(\alpha(\alpha^{-1}(f(y))-\alpha^{-1}(f(x))))}{\alpha^{-1}(\alpha(\alpha^{-1}(y)-\alpha^{-1}(x)))}\right)\\
&= \alpha\left(\frac{\alpha^{-1}(f(y))-\alpha^{-1}(f(x))}{\alpha^{-1}(y)-\alpha^{-1}(x)}\right)\\
&= \alpha\left(\frac{\alpha^{-1}(f(y))-\alpha^{-1}(f(x))}{y-x}\cdot\frac{y-x}{\alpha^{-1}(y)-\alpha^{-1}(x)}\right)\\
&\to \alpha\left(\frac{\alpha^{-1}(f(x))'}{\alpha^{-1}(x)'}\right).
\end{aligned}$$

In a similar way, informally we have

$$\begin{aligned}
\bigoplus_{i=1}^{n} f(c_i)\otimes(x_i\ominus x_{i-1}) &= \alpha\left(\sum_{i=1}^{n}\alpha^{-1}(f(c_i))(\alpha^{-1}(x_i)-\alpha^{-1}(x_{i-1}))\right)\\
&= \alpha\left(\sum_{i=1}^{n}\alpha^{-1}(f(c_i))(x_i-x_{i-1})\frac{\alpha^{-1}(x_i)-\alpha^{-1}(x_{i-1})}{x_i-x_{i-1}}\right)\\
&\to \alpha\left(\int_a^b \alpha^{-1}(f(x))\alpha^{-1}(x)'\,dx\right).
\end{aligned}$$

Therefore, the tanh-integral is defined by

$$\int_a^b f(x)\,d^\star x = \alpha\left(\int_a^b \alpha^{-1}(f(x))\alpha^{-1}(x)'\,dx\right) = \frac{e^{2\int_a^b \ln\frac{1+f(x)}{(1-f(x))(1-x^2)}dx}-1}{e^{2\int_a^b \ln\frac{1+f(x)}{(1-f(x))(1-x^2)}dx}+1}. \tag{4}$$

A removal of $\alpha^{-1}(x)'$ in (**??**) and (**??**), produces another simplified modification of tanh-derivative and tanh-integral.

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