On the spectral properties of some class of non-selfadjoint operators

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Abstract:

This report is devoted to exploration of some class of non-selfadjoint operators acting in a complex separable Hilbert space. We consider a perturbation of non-selfadjoint operator by so called "lower term" which is also a non-selfadjoint operator. In opposite to approach that was used in [1], where spectral properties of perturbation of self-adjoint and normal operators were studied, our considerations are founded on known spectral properties of the real component of non-selfadjoint operators. Having used the technic of sesquilinear forms theory we establish a compactness property of the resolvent, obtain asymptotic equivalence between the real component of resolvent and the resolvent of real component of non-selfadjoint operators. We conduct a classification of non-selfadjoint operators by belonging of their resolvent to the class of Schatten-von Neumann and formulate a sufficient condition for completeness of root vectors. Finally we obtain an asymptotic formula for the eigenvalues.

Consider the pair of complex separable Hilbert spaces $\mathfrak{H}_0, \mathfrak{H}_+$ with assumptions that $\mathfrak{H}_+$ is dense in $\mathfrak{H}_0$ as a set of elements and we have a bounded embedding. Moreover any set of elements bounded in the sense of norm $\mathfrak{H}_+$ is compact in the sense of norm $\mathfrak{H}_0$. In further we use shorthand notations for inner products and norms like in this case $(\cdot\,,\cdot)_{\mathfrak{H}_0} = (\cdot\,,\cdot)_0$. We consider the perturbation of the non-selfadjoint operator T represented by expression $W = T + A$ with certain assumptions relative to so-called "main part" - operator T and lower term - operator A, both of these operators act in $\mathfrak{H}_0$. We suppose that there exists a linear manifold dense in $\mathfrak{H}_+$ on which operators T, A are well defined with their conjugate operators. We assume that $\mathfrak{D}(W) \subset \mathfrak{D}(W^*)$. Denote the formal conjugate operator W^+ as a constriction of the operator W^* on the set $\mathfrak{D}(W)$. The denotation $\tilde{W}^+$ means closure of the operator W^+. Denote as θ the half angle of sector containing the numerical range of values of the operator $\tilde{W}$ with top situated at the point zero. Denote positive real constants as $C, C_i,\ i \in \mathbb{N}_0$. We suppose that the following conditions hold relative to the operators T, A considered above

$$\text{i)}\ \mathrm{Re}(Tf,f)_0 \geq C_0\|f\|_+^2,\ |(Tf,g)_0| \leq C_1\|f\|_+ \cdot \|g\|_+,\ f,g \in \mathfrak{D}(T);$$

$$\text{ii)}\mathrm{Re}(Af,f)_0 \geq C_2\|f\|_0^2,\ \|Af\|_0 \leq C_3\|f\|_+,\ f \in \mathfrak{D}(A).$$

We denote as H the closure of real component of the operator W. Symbol $R_H := R_H(0)$ denotes the resolvent of operator H at the point zero. Denote as $\mu := \mu(H)$ the order of operator H if we have an estimate $\lambda_n(R_H) \leq C\, n^{-\mu},\ n \in \mathbb{N},\ 0 \leq \mu < \infty$.

The following series of theorems are formulated in terms of the order of operator H and devoted to Schatten-von Neumann's classification.

Theorem 0.1. *The condition $\mu p > 2$ is sufficient for inclusion $R_{\tilde{W}} \in \mathfrak{S}_p, 1 < p < \infty$, condition $\mu > 1$ is sufficient for inclusion $R_{\tilde{W}} \in \mathfrak{S}_1$. Moreover under assumption $\lambda_n(R_H) \geq C_4\, n^{-\mu},\ n \in \mathbb{N}$, the condition $\mu p > 1$ is necessary for inclusion $R_{\tilde{W}} \in \mathfrak{S}_p, 1 \leq p < \infty$.*

The following theorem establishes the completeness property of root vectors of operator $R_{\tilde{W}}$.

Theorem 0.2. *If condition $\theta < \pi\mu/2$ is fulfilled, then the system of root functions of operator $R_{\tilde{W}}$ is complete in the space $\mathfrak{H}_0$.*

We can get asymptotic formula for the eigenvalues of operator $R_{\tilde{W}}$, if we have information about the order of operator H. This idea is realized in the following theorem.

Theorem 0.3. *If $1 \leq p < \infty$, then the following relation holds*

$$\sum_{i=1}^{n} |\lambda_i(R_{\tilde{W}})|^p \leq \sec^p \theta \sum_{i=1}^{n} \lambda_i^p(R_H),\ n = 1, 2, \ldots, \nu(R_{\tilde{W}}),$$

where $\nu(R_{\tilde{W}})$ is the sum of all algebraic multiplicities of the operator $R_{\tilde{W}}$. Moreover if $\nu(R_{\tilde{W}}) = \infty$, $\mu \neq 0$, then the following asymptotic formula holds

$$|\lambda_i(R_{\tilde{W}})| = o\left(i^{-\mu+\varepsilon}\right),\ i \to \infty,\ \forall \varepsilon > 0.$$

Keywords: Non-selfadjoint operator, perturbation of operator, real component of operator, asymptotic formula, sectorial property, accretive property, regular accretive property.

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