A study on Heron triangles and difference equations

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Abstract: In this paper, we investigate the relationship between Heron triangles formed by consecutive ordered triple $(un - v, un, un + v) \in \mathbb{Q}^3$ and Difference Equation where u, n, v are rational numbers.

Keywords: Heron Triangle, Difference Equation, Consecutive Trios.

Linear equations with two variables root of real numbers indicates a line. However find the integer roots of same equation requires a completely different approach. Similar to obtain the length of the side and the area which are triangles with rational of the family of triangles requires a different method. This paper produces an infinite family of triangles with rational sides and rational areas and, in fact, an infinite family of triangles with integer sides and integer areas. The results are quite attractive and open a door to a further study of these triangles, both geometrically and number-theoretically.

In [1], Buchholz and Rathbun (1997) studied an infinite set of Heron triangles with two rational medians. In [2], Buchholz and Rathbun (1998) presented a proof that there exist infinitely many rational sided triangles with two rational medians and rational area. These triangles correspond to rational points on an elliptic curve of rank one. They also displayed three triangles (one previously unpublished), which do not belong to any of the known infinite families. In [3], Sastry applied a technique in his study that uses a triple of integers that can represent the lengths of the sides of a triangle to generate a family of Heron triangles. In [7], Chisholm and MacDougall examined the remaining six configurations, which were left unsolved by Buchholz [R.H. Buchholz, Perfect pyramids, Bull. Austral. Math. Soc. 45 (1991) 353–368] from the fifteen configurations for tetrahedra having integer edges and volume, by restricting attention to those with two or three different edge lengths, for integer volume, completely solving all but one of them. In [9], Ionascu et al. studied the function $H(a, b)$, which associates to every pair of positive integers a and b the number of positive integers c such that the triangle of sides a, b and c is Heron, i.e., it has integral area. In particular, they proved that $H(p, q) \leq 5$ if p and q are primes, and that $H(a, b) = 0$ for a random choice of positive integers a and b.

In this study, it is investigated the n which is rational numbers, so that obtained consecutive triples that makes a triples of Heron against each value given to v in $(un(k)-v, un(k), un(k)+v) \in \mathbb{Q}^3$ which and u, v are rational numbers, can be obtained from the equation $n(k)=\frac{1}{u}\sqrt{3m^2(k)+4^2}$, where the where $m(k)$ is derived from $m(k)=\frac{1}{\sqrt{3}}v\left\{\left(2+\sqrt{3}\right)^k-\left(2-\sqrt{3}\right)^k\right\}$, $k=0,1,2,\ldots$, which are the solutions of the linear difference equation $m(k+2)-4m(k+1)+m(k)=0$ with the initial conditions $m(0)=0$ and $m(1)=2v$.

Theorem 1. *The Heronian triangles in the form* $(un-1, un, un+1)$ *are obtained from difference equation* $m(k+2)-4m(k+1)+m(k)=0$ *, initial conditions of which are* $m(0)=\frac{2}{u}$ *and* $m(1)=\frac{4}{u}$ *. The area of such triangles are obtained from the formula* $\triangle_k=\frac{\left\{\left(2+\sqrt{3}\right)^{2k}-\left(2-\sqrt{3}\right)^{2k}\right\}}{4}$ *where* $k=0,1,2,\ldots$.

Theorem 2. *The Heronian triangles in the form* $(un-2, un, un+2)$ *are obtained from difference equation* $m(k+2)-4m(k+1)+m(k)=0$ *, initial conditions of which are* $m(0)=\frac{4}{u}$ *and* $m(1)=\frac{8}{u}$ *. The area of such triangles are obtained from the formula* $\triangle_k=\sqrt{3}\left\{\left(2+\sqrt{3}\right)^{2k}-\left(2-\sqrt{3}\right)^{2k}\right\}$ *where* $k=0,1,2,\ldots$.

Theorem 3. *The Heronian triangles in the form* $(un-v, un, un+v)$ *are obtained from difference equation* $m(k+2)-4m(k+1)+m(k)=0$ *, initial conditions of which are* $m(0)=0$ *and* $m(1)=2v$ *. The area of such triangles are obtained from the formula* $\triangle_k=\sqrt{3}\left\{\left(2+\sqrt{3}\right)^{2k}-\left(2-\sqrt{3}\right)^{2k}\right\}$ *where* $k=0,1,2,\ldots$

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